Received: *December 23, 1998*, Accepted: *January 11, 1999*

Degeneration of ALF D_n metrics

Gordon Chalmers

Argonne National Laboratory, High Energy Physics Division,
9700 South Cass Avenue, Argonne, IL 60439-4815
E-mail: chalmers@sgi2.hep.anl.gov

Martin Roček and Steven Wiles

Institute for Theoretical Physics, State University of New York,
Stony Brook, N. Y. 11794-3840
E-mail: rocek@insti.physics.sunysb.edu,
swiles@insti.physics.sunysb.edu

Abstract: Beginning with the Legendre transform construction of hyperkähler metrics, we analyze the ALF version of the D_n metrics. We determine the constraint equation obtained from extremizing the w coordinate of the generating function $F(z, \bar{z}, u, \bar{u}, w)$ and study its behavior as we send two of the mass parameters of the D_n metric to zero. We find that the constraint equation enforces the limit that the metric becomes that of multi-Taub-NUT.

Keywords: Nonperturbative Effects, Duality in Gauge Field Theories, D-branes, Differential and Algebraic Geometry.

JHEP01(1999)009

Contents

JHEP01(1999)009

1. Introduction

The ALF versions of the D_n metrics have been shown [1, 2, 3] to be the appropriate metrics for the moduli space of several interesting and related theories. Of perhaps most interest currently are their relationship to M-theory. They can be found as the metrics of M-theory pertaining to an Atiyah-Hitchin space with n Kaluza-Klein monopoles. In the M-theory compactification to type IIA string theory, this corresponds to a vacuum with an orientifold 6-plane and n parallel D6-branes, respectively. A type IIB interpretation is given in [3]. They may also be found as the metric of the moduli space of $d = 2 + 1$ $N = 4$ supersymmetric $SU(2)$ Yang-Mills theory with n fundamental hypermultiplets. These two apparently different spaces are related. If one probes the background of the type IIA string theory described above with a D2-brane parallel to the D6-branes, the resulting low-energy theory on the D2-brane is that of $N = 4$ SYM with n hypermultiplets. The theory has as moduli the 3 scalars of the vector multiplet in $2+1$ dimensions, as well as the dual photon associated with the $U(1)$ of the spontaneously broken $SU(2)$. It also has as parameters the 3 masses associated with each hypermultiplet. These masses correspond in the compactified M-theory to the positions of each corresponding brane in the transverse space. These metrics also appear in other contexts, such as spontaneously broken $N = 4$ $U(N)$ SYM theory [4, 5, 6], and in monopole dynamics problems [7].

It is known [1] that the appropriate metric on M-theory with one compact direction and n KK monopoles is a "superposition" of the Atiyah-Hitchin metric and the metric of Euclidean multi-Taub-NUT space [8]. In the type IIA language, if a brane is taken to infinity, or in the SYM language if the mass of the hypermultiplet is taken to infinity, then it decouples from the theory and the metric becomes D_{n-1} ALF metric. It was shown using QFT arguments [9] that if one places two of the branes onto the orientifold plane (*i.e.*,lets two masses vanish), the metric becomes purely that of multi-Taub-Nut. In this paper, we try to see how this behavior can arise geometrically from the constraint equation of the Legendre transform construction [12]. The form of the multi-Taub-NUT metrics we find have an additional Z_2 symmetry that combines with the cyclic C_k group to produce the dihedral group of the D_k series (i.e. the metric is unchanged by $\vec{r} \to -\vec{r}$).

We begin in section 2 by reviewing the Legendre transform construction for the general $\mathcal{O}(2)$ and $\mathcal{O}(4)$ construction. In section 3, we derive and discuss the constraint equation for the Kahler potential of the D_n metrics in the Legendre transform construction. In section 4, we examine the behavior of this constraint under the limit and see how it enforces the condition on the metric $g_{D_n} \to g_{mTN_{2n}}$. Finally, in section 5 we discuss open problems and related issues.

2. The $\mathcal{O}(2)$ and $\mathcal{O}(4)$ Legendre transform constructions

In this section we briefly review the generalized Legendre tranform construction [11]. We first consider the $\mathcal{O}(2)$ case. We begin by defining ζ to be the coordinate on the Riemann sphere. Complex conjugation on the sphere is defined by the antipodal mapping $\zeta \to -1/\bar{\zeta}$.

In the Legendre construction we define a 2^{nd}-order polynomial $\eta(\zeta) = \sum_a^{n=2} w_a \zeta^a$, i.e. a section of a $\mathcal{O}(2)$ line bundle over the sphere. It obeys the reality condition $\overline{\eta(-1/\bar{\zeta})} = -\zeta^{-2}\eta(\zeta)$, which implies that $w_{2-a} = -(-1)^a \bar{w}_a$. This constrains its form to be $\eta(\zeta) = \bar{z} + x\zeta - z\zeta^2$, where z is a complex coordinate and x is a real coordinate.

We define a function F as follows

$$F(z, \bar{z}, v, \bar{v}, x) = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta^2} G(\eta, \zeta)\,, \tag{2.1}$$

where $G(\eta(\zeta), \zeta)$ is some arbitrary function of the two variables ζ and η and C is an appropriately chosen contour.

The Kahler potential $\mathcal{K}(z, \bar{z}, u, \bar{u})$ is constructed from F by means of the Legendre transformation

$$\mathcal{K}(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, x) - (u + \bar{u})x\,, \tag{2.2}$$

where $x(z, \bar{z}, u, \bar{u})$ is a solution to the minimization

$$\frac{\partial F}{\partial x} = u + \bar{u} \tag{2.3}$$

The metric may then be constructed as usual from the Kahler potential in terms of the coordinates $z, \bar{z}, u, \bar{u}$ by the appropriate second derivatives [11].

As an example relevant to this paper, we construct of the multi-Taub-NUT metrics. In this case

$$
\begin{aligned}
F = & -\oint_{C_0} \frac{d\zeta}{2\pi i \zeta} \frac{\eta^2(\zeta)}{\zeta^2} + \oint_{C'} \frac{d\zeta}{\zeta} \frac{\eta(\zeta)\ln(\zeta)}{\zeta} - \\
& - \sum_{i=1}^{n} \sum_{+,-} \oint_{C_i} \frac{d\zeta}{2\pi i \zeta^2} \left(\eta(\zeta) \pm \chi_i(\zeta)\right) \ln\left(\eta(\zeta) \pm \chi_i(\zeta)\right) .
\end{aligned} \tag{2.4}
$$

Here $\chi(\zeta)_i$ are $\mathcal{O}(2)$ polynomials with constant coefficients. The contour C' is a lemniscate enclosing the two roots of $\eta(\zeta)$, while the C_i are lemniscates enclosing the roots of $\eta(\zeta) \pm \chi_i(\zeta)$. The contour C_0 is a simple contour enclosing the origin. All $4k$-dimensional metrics admitting k tri-holomorphic isometries may be constructed in terms of k independent $O(2)$ multiplets [10].

Other metrics may be constructed in terms of higher order polynomials $O(2n)$ satisfying a reality constraint [11]. In the $\mathcal{O}(4)$ case, we define a polynomial on the sphere $\eta(\zeta) = \sum_a^{n=4} w_a \zeta^a$. We require this polynomial to obey the reality condition $\overline{\eta(-1/\bar{\zeta})} = \zeta^{-4}\eta(\zeta)$, which implies $w_{4-a} = (-1)^a \bar{w}_a$. These conditions allow us to write the polynomial as $\eta(\zeta) = \bar{z} + \bar{v}\zeta + x\zeta^2 - v\zeta^3 + z\zeta^4$, where z and v are complex and x is real.

As before (2.1), we define a function F and perform the following Legendre transform to obtain the Kahler potential

$$
\mathcal{K}(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, x) - uv - \bar{u}\bar{v} \,, \tag{2.5}
$$

where $v(z, \bar{z}, u, \bar{u})$; in this case we must eliminate the auxiliary coordinate $x(z, \bar{z}, u, \bar{u})$ via the solution to

$$
\begin{aligned}
\frac{\partial F}{\partial v} &= u \,, \\
\frac{\partial F}{\partial \bar{v}} &= \bar{u} \,, \\
\frac{\partial F}{\partial x} &= 0 \,.
\end{aligned} \tag{2.6}
$$

Solving this constraint equation is difficult in practice, as generally one obtains transcendental equations (in the coordinates chosen above).

Without solving for the constraint equation, the construction of a particular metric on a given hyperkähler manifold becomes the question how to find the proper function G and contour C, *i.e.*, the function F. The metrics we wish to study were determined in [3] using twistor methods and studied previously in [12] and [2] using twistor space methods.

In [9], it was shown that if two hypermultiplet masses vanish in the D_n metrics for $n > 2$, then the metric becomes a multi-Taub-Nut that describes the perturbative correction to the low-energy theory on the brane (the instanton contributions vanish). Physically, this is relatively easy to understand in the M-theory language as the cancelling of the Ramond charges of the two 6-branes with that of the orientifold 6-plane as they approach each other, flattening the Atiyah-Hitchin metric and leaving only the Taub-Nut metric of the remaining branes. In the Legendre construction language, how this phenomenon can arise is much more mysterious. The D_n metrics are constructed from the $\mathcal{O}(4)$ construction, whereas the multi-Taub-Nut metrics are constructed from the $\mathcal{O}(2)$ construction, which have different structures (for example, the latter has a triholomorphic isometry). How the vanishing of two of the masses in the constraint equation of the $\mathcal{O}(4)$ construction could transform the Kahler potential into one derivable from an $\mathcal{O}(2)$ construction is not self-evident. In this note, we will show how the limiting behavior of the constraint equation results in precisely such behavior.

3. The constraint equation

The generating functional for the ALF-D_n metrics was suggested and supported originally in [12, 2] and was later proven in [3] to be

$$\begin{aligned} F = &-\oint_{C_0} \frac{d\zeta}{2\pi i\zeta}\frac{\eta(\zeta)}{\zeta^2} + \oint_C \frac{d\zeta}{\zeta}\frac{\sqrt{\eta(\zeta)}}{\zeta} - \\ &-\sum_{i=1}^{n}\sum_{+,-}\oint_{C_i}\frac{d\zeta}{2\pi i\zeta^2}\left(\sqrt{\eta(\zeta)} \pm \chi_i(\zeta)\right)\ln\left(\sqrt{\eta(\zeta)} \pm \chi_i(\zeta)\right). \end{aligned} \tag{3.1}$$

The χ_i are $\mathcal{O}(2)$ polynomials in ζ, parameterizing the deformations in the D_k ALF metric. The metric is unchanged by reversing the sign of the deformation parameters $\chi_i \to -\chi_i$. They obey the reality condition defined in the previous section for $\mathcal{O}(2)$ polynomials and can expressed as $\chi_i(\zeta) = \bar{p}_i + q_i\zeta - p_i\zeta^2$, where p_i are complex parameters and q_i are real parameters. They parametrize the position of the monopoles (branes) in the theory, or equivalently, the masses of the hypermultiplet.

Because of the reality condition, the roots of $\eta(\zeta)$ always come in pairs $\alpha, -1/\bar{\alpha}$ and $\beta, -1/\bar{\beta}$. The contour integrals are easier to work with if we rewrite η as follows

$$\eta(\zeta) = \rho\left(\zeta - \alpha\right)\left(\zeta - \beta\right)\left(\bar{\alpha}\zeta + 1\right)\left(\bar{\beta}\zeta + 1\right), \tag{3.2}$$

where ρ is real and independent of ζ. We similarly rewrite each χ_i in terms of a real scale σ_i and a complex root r_i as follows

$$\chi_i(\zeta) = \sigma_i(\zeta - r_i)(\bar{r}_i\zeta + 1). \tag{3.3}$$

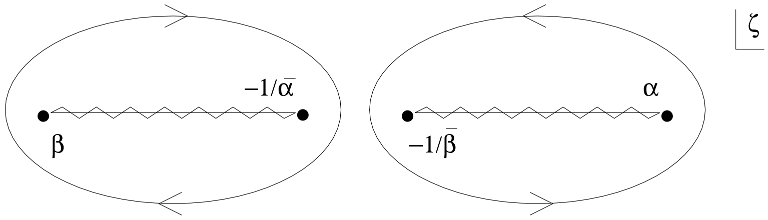

Figure 1: The contour C.

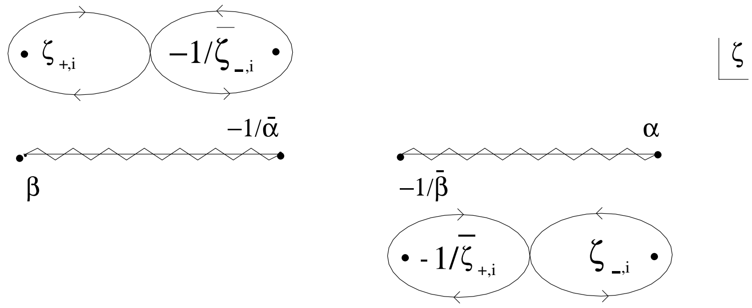

Figure 2: The contour C_i.

The contour C_0 is defined as a simple contour enclosing the ζ origin counter-clockwise. The contour C is defined as two simple contours around the branch cuts of $\sqrt{\eta(\zeta)}$ as shown in figure 1.

The contours C_i are represented in figure 2, where the lemniscates are defined to encircle the roots of $\sqrt{\eta} - \chi_i$ and $\sqrt{\eta} + \chi_i$, labelled above as $\zeta_{+,i}, -1/\bar{\zeta}_{-,i}$ and $\zeta_{-,i}, -1/\bar{\zeta}_{+,i}$ respectively. The orientation is chosen so that when these roots correspond to the roots of $\eta(\zeta)$, they exactly cancel the contribution from C.

By the Legendre transform construction outlined previously, we must extremize the generating function with respect to the coordinate x. The resulting constraint equation is

$$\frac{\partial F}{\partial x} = 0 = -\oint_{C_0} \frac{d\zeta}{2\pi i \zeta} + \frac{1}{2} \oint_C \frac{d\zeta}{\sqrt{\eta(\zeta)}} - \sum_{i=1}^{n} \oint_{C_i} \frac{d\zeta}{\sqrt{\eta(\zeta)}} \ln\left(\eta(\zeta) - \chi_i(\zeta)^2\right). \tag{3.4}$$

The first contour integral gives -1. The other contour integrals are expressible as a sum of complete and incomplete elliptic integrals and yield the following constraint equation

$$0 = -1 + \frac{(4-2n)K(k) + 2\sum_{i=1}^{n} Re\left(F(\phi_i, k) + F(\psi_i, k)\right)}{\sqrt{\rho(1+\alpha\bar{\alpha})(1+\beta\bar{\beta})}}, \tag{3.5}$$

where $K(k)$ is the complete elliptic integral of the first kind and $F(x,k)$ is the incomplete elliptic integral of the first kind.[1] The arguments of the incomplete elliptic

[1]The definition of the incomplete elliptic integral of the first kind $F(x,k)$ which we will use is,

integrals are

$$\phi_i = \sqrt{\frac{(1+\beta\bar{\beta})(a_i-\alpha)}{(1+\beta\bar{\alpha})(a_i-\beta)}}\,, \qquad \psi_i = \sqrt{\frac{(1+\alpha\bar{\alpha})(b_i-\beta)}{(1+\alpha\bar{\beta})(b_i-\alpha)}} \tag{3.7}$$

and the modulus is

$$k = \sqrt{\frac{(1+\alpha\bar{\beta})(1+\beta\bar{\alpha})}{(1+\alpha\bar{\alpha})(1+\beta\bar{\beta})}}\,. \tag{3.8}$$

Here a_i and b_i are respectively the two roots of the equation $\eta - \chi_i^2 = 0$ analytically connected to the original roots α and β, and are considered to be implicit functions of α,β, ρ, σ_i, and r_i.

As in [12], it is most convenient to attempt to solve for this equation in terms of the variable ρ rather than the original variable x, resulting in the final form of the constraint equation we work with

$$\sqrt{\rho} = \frac{(4-2n)K(k) + 2\sum_{i=1}^{n} Re(F(\phi_i,k)+F(\psi_i,k))}{\sqrt{(1+\alpha\bar{\alpha})(1+\beta\bar{\beta})}}\,. \tag{3.9}$$

Unfortunately, as the variables a_i and b_i are implicitly functions of the variable ρ, we cannot obtain an analytic solution for ρ: The ϕ_i and ψ_i are functions of ρ through equation (3.7) and the solution for ρ in this set of coordinates is transcendental.

4. Behavior of the constraint equation

To begin our study of the behavior of the constraint equation, it is useful to first consider its behavior when we take any given parameter χ_i to be infinitely large. Physically, this corresponds to setting the masses of the i^{th} hypermultiplet to infinity [9], or to taking the transverse position of the i^{th} 6-brane to spatial infinity [2, 1]. We must therefore find that the resulting constraint equation is that of a system with the i^{th} monopole (6-brane) removed, *i.e.*, we must find that

$$Re\,(F(\phi_i,k)+F(\psi_i,k)) \to 2K(k) \tag{4.1}$$

where the left-hand side approaches $K(k)$ monotonically from below.[2]

---

for real values of the argument x and the modulus k,

$$F(x,k) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,. \tag{3.6}$$

Since the arguments which appear in the constraint equation for this function are in general complex, we take the analytic continuation of $F(x,k)$ to complex values of x. The modulus k is always real in our equations. The complete elliptic integral is defined as $K(k) = F(1,k)$.

[2] The limit must always be from below or there would be some finite distance (mass) for which a brane (hypermultiplet) completely decoupled from the theory, which is not physically reasonable.

JHEP01(1999)009

For simplicity, consider taking the limit by sending σ_i to infinity. It is clear that as $\sigma_i \to \infty$, the roots of $\eta - \chi_i^2$ approach those of χ_i. For real roots α, β and r_i, it is easy enough to use the addition properties of incomplete elliptic integral to prove (4.1), and the result analytically continues. One may, of course, continue to decouple all of the monopoles, which will leave the constraint equation for the Atiyah-Hitchin space [12].

The asymptotic regime occurs when the $\mathcal{O}(4)$ coordinate degenerates into $\mathcal{O}(2)^2$. The metric is a double cover of (the singular) multi-Taub-Nut. The latter is described by eq.(2.4). The metric may be easily derived.

The behavior of the D_n metrics presented in [9, 1] is that for $n > 2$, if one sets two of the χ_i to zero, the metric of the moduli space should be the multi-Taub-Nut metric [1]. The asymptotic form of the D_k ALF metric is,

$$ds^2 = V(\vec{r}, \vec{r}_i) d\vec{r} \cdot d\vec{r} + V^{-1}(\vec{r}, \vec{r}_i)(d\phi + \vec{A} \cdot d\vec{r})^2 \,, \tag{4.2}$$

where $r = \sqrt{x^2 + 4z\bar{z}}$ (*cf.* eq. (2.4)),

$$V(\vec{r}, \vec{r}_i) = 1 - \frac{4}{|\vec{r}|} + \sum_{k=1}^{n} \frac{1}{|\vec{r} - \vec{r}_i|} + \frac{1}{|\vec{r} + \vec{r}_i|} \,, \tag{4.3}$$

and $\vec{\nabla} \times \vec{A} = \vec{\nabla} V$. The form of this metric is singular at some finite radius where $V = 0$ (of course, this singularity is not present in the full D_k ALF metric, just in the asysmptotic form). However, when we take two of the $\vec{r}_i$ to zero, the contribution cancels the negative term in V and forces the net result to be smooth.

In the language of the Legendre tranform construction, this corresponds to the constraint equation enforcing the condition $\eta_4(\zeta) \to \tilde{\eta}_2(\zeta)^2$ for *fixed* ρ, where $\tilde{\eta}_2$ is an $\mathcal{O}(2)$ polynomial of ζ. Unfortunately, it is not possible to prove this by setting two χ_i to zero identically. If one does so, the corresponding roots of $\eta - \chi_i^2 = 0$ are simply $a_i = \alpha$ and $b_i = \beta$ and the corresponding incomplete elliptic integrals vanish. The incomplete elliptic integral terms for the remaining χ_i, however, may never exceed $(2n-4)K(k)$ in value combined, and so the right-hand side of the constraint equation is always negative. Thus, the only solution to the constraint equation for any values of α, β, and the $n-2$ remaining χ_i necessarily must be a negative value of $\sqrt{\rho}$. Such a value seems unphysical for reasons we discuss later, and we ignore such solutions.[3] Note, however, that negative values of $\sqrt{\rho}$ give the same four solutions as positive

[3] Each point in (z, v, x) space uniquely defined a given polynomial $\eta(\zeta)$. However, a point in (ρ, α, β) space only uniquely defines η up to the set of permutations of the roots with appropriate rescalings of ρ. These appear as discrete isometries in the metric. The permutation $\alpha \to -1/\bar{\alpha}, \rho \to -\rho/\alpha\bar{\alpha}$ allows us to consider only positive values of ρ. The metric also has a $U(1)$ non-triholomorphic isometry that implies $F(\rho, \alpha, \beta, \sigma_i, r_i) = F(\rho, e^{i\theta}\alpha, e^{i\theta}\beta, \sigma_i, e^{i\theta} r_i)$, and which we use to consider only real values of α in the rest of this paper [13, 2].

JHEP01(1999)009

values in (4.5), up to permutations. We must take more care and instead consider a limiting procedure for taking two χ_i to zero.

Since the a_i and b_i are implicit functions of $\sqrt{\rho}$, how can we best analyze this limit? Consider the following. The variables a_i and b_i are by definition roots of the equation

$$\rho(\zeta-\alpha)(\zeta-\beta)(\bar{\alpha}\zeta+1)(\bar{\beta}\zeta+1)-\sigma_i^2(\zeta-r_i)^2(\bar{r}_i\zeta+1)^2=0\,, \tag{4.4}$$

which in turn means they are also roots of the equation

$$(\zeta-\alpha)(\zeta-\beta)(\bar{\alpha}\zeta+1)(\bar{\beta}\zeta+1)-\left(\frac{\sigma_i}{\sqrt{\rho}}\right)^2(\zeta-r_i)^2(\bar{r}_i\zeta+1)^2=0\,. \tag{4.5}$$

It is easiest to study the behavior of the constraint equation by scaling out ρ. We use the following procedure: Assume α and β are fixed. Fix all ratios of the σ_i with respect to each other and vary all χ_i by an overall scale S. To this end, we define a set of "scaled" mass parameters $\tilde{\chi}_i$ as follows

$$\tilde{\chi}_i=\tilde{\sigma}_i(\zeta-r_i)(\bar{r}_i\zeta+1)\,,\qquad i=1\ldots n \tag{4.6}$$

where $\frac{\tilde{\sigma}_i}{\tilde{\sigma}_j}=\frac{\sigma_i}{\sigma_j}$ for all $i,j=1..n$ and the smallest scale is equal to one. We now find roots a_i, b_i for the equation $\eta|_{\rho=1}-\tilde{\chi}_i^2=0$. In this approach, the right-hand side of the constraint equation no longer implicitly depends on ρ. Once we have found a solution of $\sqrt{\rho}$ for a given S, we may "rescale" $\tilde{\sigma}_i$ by and so find a consistent solution of the constraint for fixed values of α,β and parameters $\chi_i=\sqrt{\rho}S\tilde{\chi}_i$. Finding a solution for particular values of χ_i then becomes a matter of numerical iteration.

Consider the cases when $n>2$. When the scale S is set to zero, all of the incomplete elliptic integral are also zero, and the solution to the constraint equation is $\sqrt{\rho}=\frac{(4-2n)K(k)}{\sqrt{(1+\alpha\bar{\alpha})(1+\beta\bar{\beta})}}$, a negative number. As S approaches infinity, all of the monopoles will steadily decouple and the solution to the constraint equation approaches that of the Atiyah-Hitchin metric, $\sqrt{\rho}\to\frac{4K(k)}{\sqrt{(1+\alpha\bar{\alpha})(1+\beta\bar{\beta})}}$, a positive number. As the right-hand side of the constraint equation is a continuous function of its arguments, there is some $S=S_0$ for which the solution is $\sqrt{\rho}=0$. For larger values of S, the value of $\sqrt{\rho}$ is positive and steadily increases to the fully decoupled value. Thus, we find that we have consistent solutions for $\chi=0$ at more than one point, namely $S=0$ and $S=S_0$. The existence of such double solutions holds for all but one of the non-positive values of $\sqrt{\rho}$.

The graph in figure 3 represents all consistent solutions of the constraint equation (3.9) obtainable by scaling all three mass parameters χ_i with fixed ratios for an $n=3$ case. This graph is representative of the dependence of $\sqrt{\rho}$ on the true scale of an arbitrary mass parameter χ for all choices of α,β, and χ_i, and we did not include

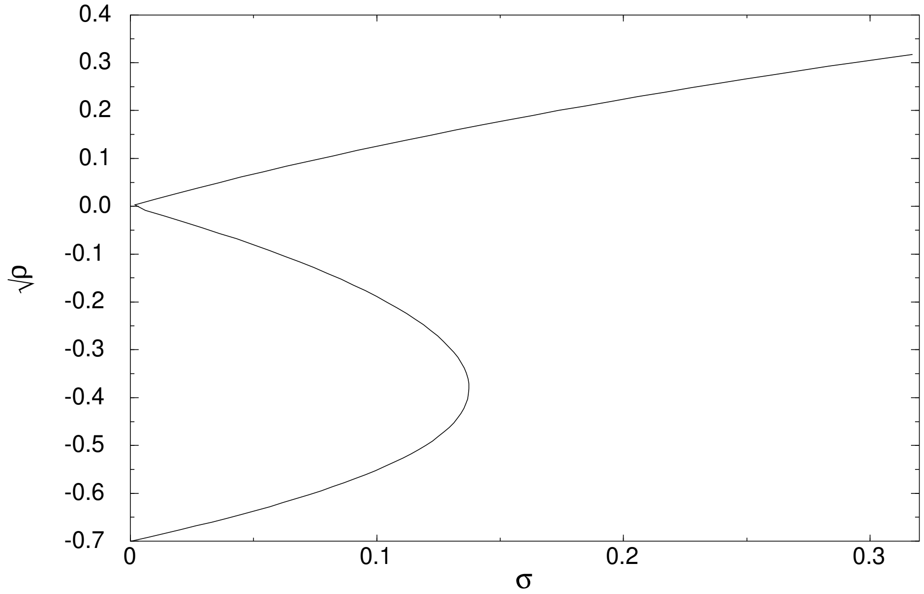

Figure 3: All consistent solutions of the constraint equation (3.9).

specific values of χ_i for simplicity. Having found multiple solutions of ρ for the same mass parameters χ_i in the $\sqrt{\rho} \leq 0$ region as well as an apparent maximum scale for the mass parameters past which no solutions exist, we do not consider this region to be physically meaningful. For completeness, we have described these "extra" solutions, but we do not believe they have any physical significance.

To analyze the limit of taking two χ_i to zero for $\sqrt{\rho}$ positive while holding $\sqrt{\rho}$ and the other χ_i fixed, consider the following procedure. Hold α, β, and all ratios $\frac{\sigma_i}{\sigma_j}, i, j = 3 \ldots n$ fixed. Define

$$\tilde{\chi}_i = s(\zeta - r_i)(\bar{r}_i \zeta + 1)\,, \qquad i = 1, 2\,, \tag{4.7}$$

where $s \ll 0$ is assumed to be a small real parameter we take to zero. It is then always possible by our previous arguments to find a particular value of $S > S_0$ such that $\sqrt{\rho} S \tilde{\sigma}_i = \sigma_i,\ i = 3 \ldots n$. By varying the value of s incrementally to zero (without reaching zero, of course) one may plot the behavior of $\sqrt{\rho}$ as a function of s for fixed values of α, β. We plotted several families of graphs for $n = 3$ with various fixed values of $r_i, i = 1 \ldots n$ and $\sigma_j, j = 3 \ldots n$ and with values of β steadily approaching a fixed real α in each family. Three such representative families of graphs are shown below.

In the two families of graphs of figures 4 and 5 the value of β is varied along the circle of radius $|\alpha|$ from graph to graph with α fixed. The legend indicates $|\alpha - \beta|$ for each graph, as well as the roots and (relative) scales of the χ_i. One value of β widely separated from α as well as three with β closely approaching α are shown for each family.

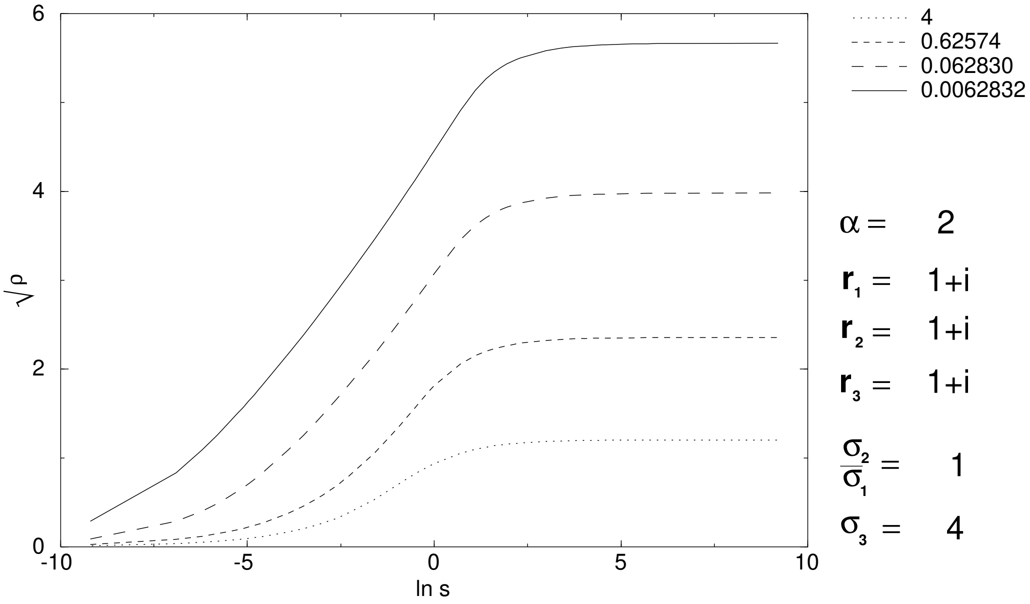

Figure 4: The value of β is varied along the circle of radius $|\alpha|$ from graph to graph with α fixed.

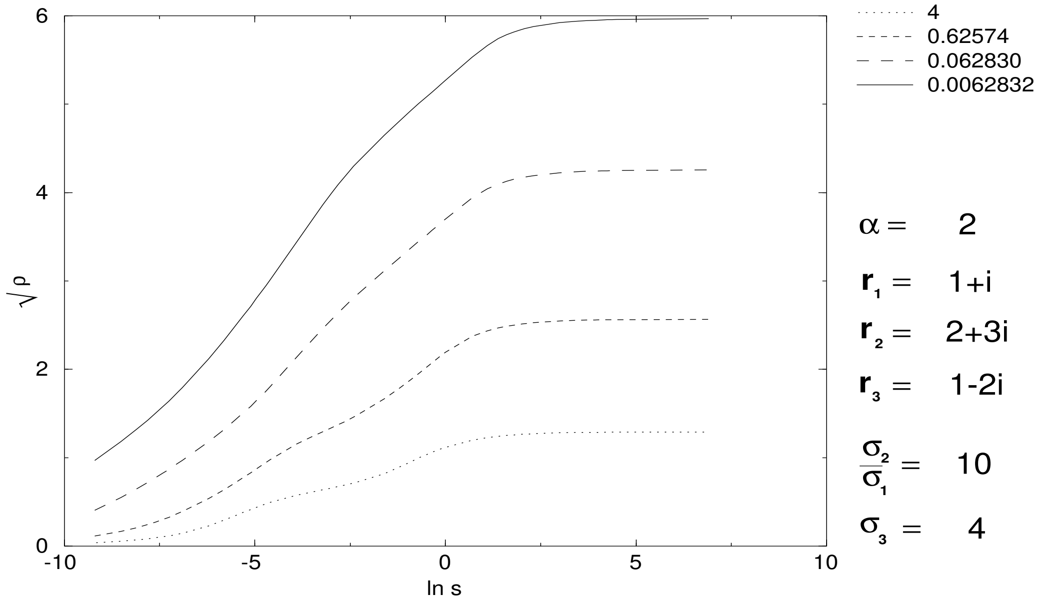

Figure 5: The value of β is varied along the circle of radius $|\alpha|$ from graph to graph with α fixed.

In the graph of figure 6, β approaches α along the real axis (from below).

It is easy to see from these graphs that as $s \to 0$, one must take values of β ever closer to α to find a solution for a given value of $\sqrt{\rho}$. Of the other twenty-four graphs which we plotted for various χ_i, each displays the same qualitative behavior.

Studying the behavior of the graphs as $s \to 0$ for fixed $\sqrt{\rho}$, one finds that the

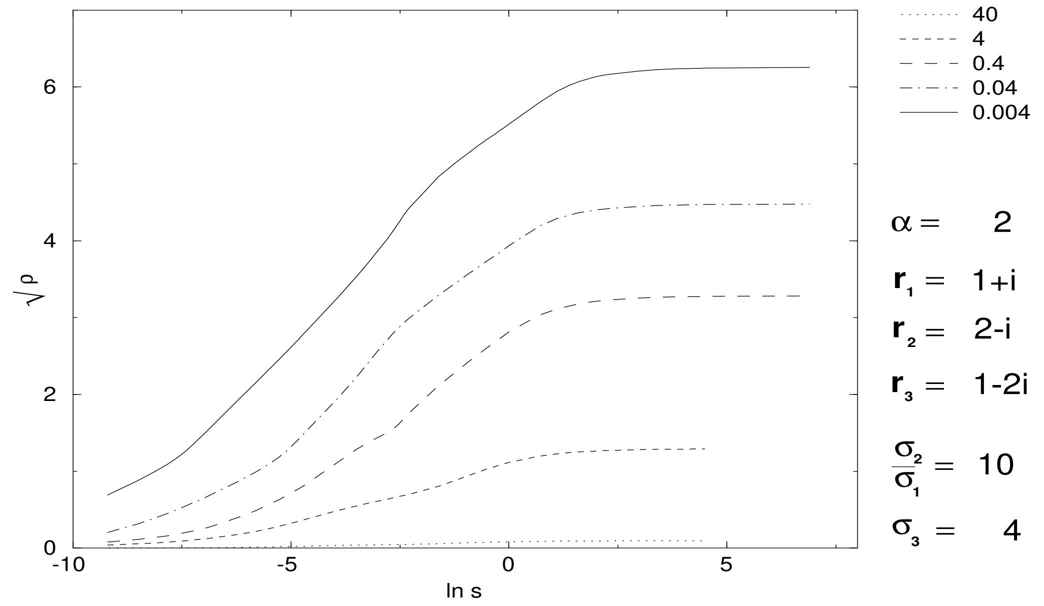

Figure 6: β approaches α along the real axis.

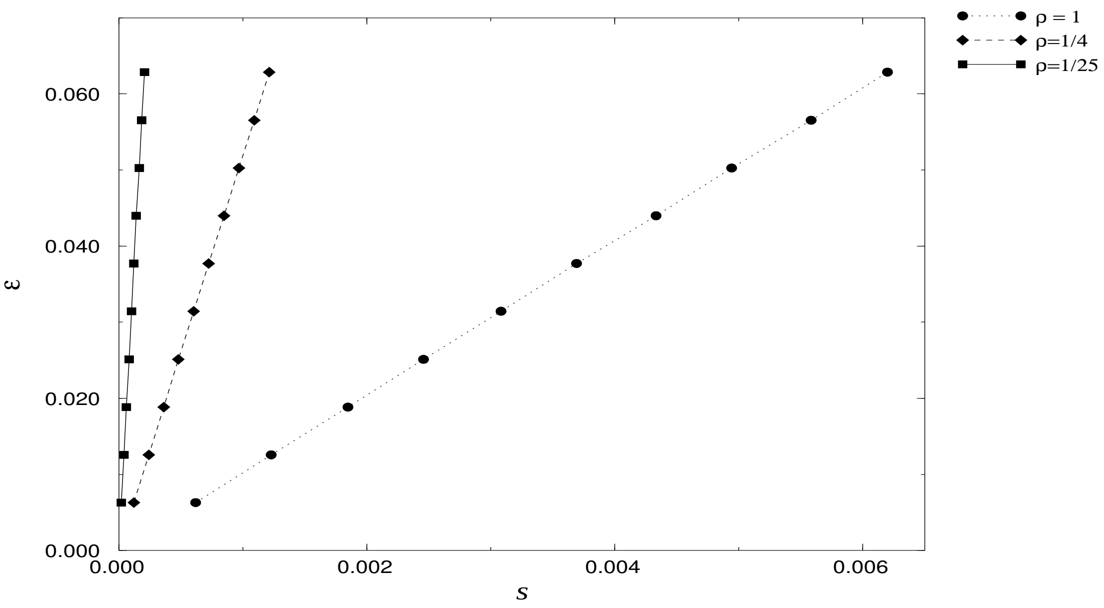

Figure 7: $|\alpha - \beta|$ is linearly related to the scale s to lowest order.

absolute difference between α and β must be linearly related to the scale s to lowest order when s and $\epsilon = |\alpha - \beta|$ are small. In figure 7 is given an example graph of this relationship for the case where $r_1 = 1 + i, r_2 = 2 + 3i, r_3 = 1 - 2i$,$\sigma_3 = 4$, and $\alpha = 2$

This graph makes it clear that the constraint equation's intrinsic behavior enforces the condition that as two mass parameters approach zero, then $|\beta - \alpha| \to 0$ for a fixed ρ, and hence $\eta_4 \to \tilde{\eta}_2^2$. (Taking more than two masses to zero will also force the coincidence of α and β for fixed ρ.)

We end this section with some implications of the above degeneration. The asymptotic form of the metric in (3.1) we are considering has the explicit form in (4.3) in the normalization of the metric chosen in eq. (2.6). This metric has the form of the one-loop correction to the vacuum moduli space metric of an $SU(2)$ $N = 4$ gauge theory gauge theory with n_k hypermultiplets. The form verifies the predictions of [9] that the instanton contributions vanish when we take two of the mass parameters to zero.

The novel feature of the way the instanton corrections are suppressed as two masses tend to zero is thus the following: the instanton corrections are *not* explicity suppressed in the function F or in F_x, but only in the *solution* to the constraint equation $F_x = 0$.

5. Conclusions and discussion

We have shown how the D_k ALF metrics become the Taub-NUT metric as two masses tend to zero in the framework of the twistor construction of [12, 3]. The suppression of the corrections that distinguish these metrics occurs in an interesting way: in the limit as the masses tend to zero, the solution to the constraint equation that arises in the construction of the D_k metrics gets "squeezed" into the asymptotic domain where the corrections become exponentially small.

Higher dimensional hyperkähler ALF spaces analogous to the D_k spaces clearly limit in the same way to higher dimensional Taub-NUT spaces. It would be interesting to analyze the constraint equation (2.6) for other cases, *e.g.,* D_k ALE spaces.

Both the A_k and the D_k spaces are known to have descriptions as algebraic curves [14]. It would also be interesting to see how the limit arises in this description.

Acknowledgments

We would like to thank S. Cherkis for discussions. G.C. would like to thank CERN for its hospitality during the final stages of this work. The work of G.C. was supported in part by NSF Grant No. PHY 9722101 and US Dept. of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38. The work of S.W. and M.R. was supported in part by NSF Grant No. PHY 9722101.

References

[1] A. Sen, *J. High Energy Phys.* **09** (1997) 1 [hep-th/9707123].

[2] G. Chalmers, *Phys. Rev.* **D 58** (1998) 125011 [hep-th/9709082];
G. Chalmers, *Multi-monopole moduli spaces for SU(n) gauge group*, hep-th/9605182.

[3] S. Cherkis and A. Kapustin, hep-th/9803112; *Nucl. Phys.* **B 525** (1998) 215 [hep-th/9803160].

[4] G. Chalmers and A. Hanany, *Nucl. Phys.* **B 489** (1997) 223 [hep-th/9608105].

[5] A. Hanany and E. Witten, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[6] D. Diaconescu, *Nucl. Phys.* **B 503** (1997) 220 [hep-th/9608163].

[7] M.F. Atiyah and N.J. Hitchin, *The geometry and dynamics of magnetic monpoles*, Princeton NF 1988.

[8] S.W. Hawking, *Phys. Lett.* **A 60** (1977) 8.

[9] N. Seiberg and E. Witten, *Gauge Dynamics and Compactification to Three Dimensions,* hep-th/9607163.

[10] N.J. Hitchin, U. Lindstrom, A. Karlhede, M. Rocek, *Comm. Math. Phys.* **108** (1987) 535.

[11] U. Lindstrom and M. Roček, *Comm. Math. Phys.* **115** (1988) 21.

[12] I. Ivanov and M. Roček, *Comm. Math. Phys.* **182** (1996) 291 [hep-th/9512075].

[13] C.P. Boyer and F.D. Finley, *J. Math. Phys.* **23** (1982) 1126.

[14] P.B. Kronheimer, *J. Diff. Geom.* **29** (1989) 665; *J. Diff. Geom.* **29** (1989) 685.